

The scalar curvature and the biorthogonal curvature: A pinching problem

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Abstract

The famous pinching problem says that on a compact simply connected n -manifold if its sectional curvature satisfies $K_{min} > \frac{1}{4}K_{max} > 0$, then the manifold is homeomorphic to the sphere. In [8, problem 12], S. T. Yau proposed the following problem: If we replace K_{max} by the scalar curvature, can we deduce similar pinching theorems? In our present note we give an answer to this question in dimension $n = 4$.

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Let $M = M^n$ be a connected differentiable n -manifold with scalar curvature s and let K be the sectional curvature. For each $x \in M$ let's consider P_1, P_2 two mutually orthogonal 2-planes in the tangent space $T_x M$. The *biorthogonal (sectional) curvature* (see [2]) relative to P_1 and P_2 is the average

$$K^\perp(P_1, P_2) = \frac{K(P_1) + K(P_2)}{2}. \quad [1.1]$$

If $n = 4$ and P is a 2-plane in $T_x M$ we will write

$$K^\perp(P) = \frac{K(P) + K(P^\perp)}{2}, \quad [1.2]$$

where P^\perp is the orthogonal complement of P in $T_x M$. The sum of two sectional curvatures on two orthogonal planes appears in the works of W. Seeman [6] and M. H. Noronha [5]. In our presents article we consider manifolds M of dimension four and the following functions:

$$K_1^\perp(x) = \min\{K^\perp(P); P \text{ is a 2- plain in } T_x M\}, \quad [1.3]$$

$$K_3^\perp(x) = \max\{K^\perp(P); P \text{ is a 2- plain in } T_x M\}, \quad [1.4]$$

$$K_2^\perp(x) = \frac{s(x)}{4} - K_1^\perp(x) - K_3^\perp(x). \quad [1.5]$$

The biorthogonal curvature and the Weyl tensor

Let (M, g) be an oriented Riemannian 4-manifold. For each $x \in M$ the bundle of two-forms Λ^2 of M splits $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ into \pm -eigenspaces of the Hodge \star -operator: $\Lambda^\pm = \{\alpha \in \Lambda^2; \star\alpha = \pm\alpha\}$. The Weyl tensor W is an endomorphism of Λ^2 such that $W = W^+ \oplus W^-$, where $W^\pm : \Lambda^\pm \rightarrow \Lambda^\pm$ are self-adjoint with free traces and are called of the self-dual and anti-self-dual parts of W , respectively. Let $w_1^\pm \leq w_2^\pm \leq w_3^\pm$ be the eigenvalues of the tensors W^\pm , respectively. As was proved in [2],

$$K_1^\perp - \frac{s}{12} = \frac{w_1^+ + w_1^-}{2}, \quad [1.6]$$

$$K_2^\perp - \frac{s}{12} = \frac{w_2^+ + w_2^-}{2} \quad [1.7]$$

and

$$K_3^\perp - \frac{s}{12} = \frac{w_3^+ + w_3^-}{2} \quad [1.8]$$

Based on a proposed question by Yau [8, problem 12], the authores of the article [3, page 16] proposed the following conjecture

Let (M, g) be a compact simply connected Riemannian n -manifold scalar curvature $s > 0$ and sectional curvature K . If $K > \frac{s}{n(n+2)}$ on M then M is diffeomorphic to the sphere \mathbb{S}^4 .

In dimension $n = 4$ we obtained the following

Theorem 1 - *Let (M, g) be a compact oriented 4-manifold with scalar curvature $s > 0$. Let K_1^\perp and K_3^\perp be the biorthogonal curvatures given by [1.3] and [1.4], respectively. If $K_1^\perp \geq \frac{s}{24}$ on M or $K_3^\perp \leq \frac{s}{6}$ on M then we have*

(1) *M is diffeomorphic to a connected sum $\mathbb{S}^4 \# (\mathbb{R} \times \mathbb{S}^3)/G_1 \# \dots \# (\mathbb{R} \times \mathbb{S}^3)/G_n$, where the G_i are discrete subgroup of the isometry group of $\mathbb{R} \times \mathbb{S}^3$ or*

(2) *(M, g) is conformal to a complex projective space \mathbb{CP}^2 with the Fubini-Study metric or*

(3) *The universal covering of M is isometric to product $\mathbb{R} \times N^3$, where N^3 is diffeomorphic to \mathbb{S}^3 .*

Remark 1.1- Compare the above Theorem 1 to Theorem 1.1 in [3] and the Conjecture A in page 17 of [3].

Proof of Theorem 1

Let (M, g) be a compact oriented Riemannian 4-manifold. It is known that M has nonnegative isotropic curvature if $w_3^\pm \leq s/6$ (see [4]), where w_3^\pm are the largest eigenvalues of W^\pm , respectively. Equation [1.6] and the condition $K_1^\perp \geq \frac{s}{24}$ implies that $w_1^+ \geq w_1^+ + w_1^- \geq -s/12$ and so $w_3^+ = -w_1^+ - w_2^+ \leq -2w_1^+ \leq s/6$. Similarly, $w_3^- \leq s/6$ and this proves that M has nonnegative isotropic curvature. Notice that the condition $K_3^\perp \leq \frac{s}{6}$ also implies that M has nonnegative isotropic curvature. On the other hand, it is easy to see that if $M = M_1^2 \times M_2^2$ then M has $K_1^\perp = K_2^\perp = 0$ which contradicts the initial hypotheses. So, this proves that if M is reducible then the universal covering of M is isometric to product $\mathbb{R} \times N^3$, where N^3 is diffeomorphic to sphere \mathbb{S}^3 . If M is irreducible then the Theorem 1 follows from principal results in [7] and [1].

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